

Gravitational waves with torsion in 3D

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Abstract

We study gravitational waves with torsion as exact vacuum solutions of three-dimensional gravity with propagating torsion. The new solutions are a natural generalization of the plane-fronted gravitational waves in general relativity with a cosmological constant, in the presence of matter.

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I. INTRODUCTION

Investigations of three-dimensional (3D) gravity have had an important influence on our understanding of both classical and quantum aspects of the realistic gravitational dynamics. In this context, the traditional approach based on *general relativity* has led to a number of outstanding results [1]. However, in the early 1990s, Mielke and Baekler [2] initiated a new approach to 3D gravity, relying on a modern field-theoretic formulation of gravity, the *Poincaré gauge theory* (PGT), proposed in the early 1960s by Kibble and Sciama [3–6]. Compared to general relativity, dynamical structure of PGT is extended by using both the curvature and the torsion to describe the associated Riemann–Cartan (RC) geometry of spacetime.

The Mielke–Baekler model, like Einstein’s general relativity, is a topological theory without propagating degrees of freedom. In PGT, such an unrealistic feature of the gravitational dynamics can be naturally improved by going over to a Lagrangian that is at most quadratic in torsion and curvature (quadratic PGT). Recent investigations reveal elements that indicate a rich dynamical structure of the *quadratic PGT* [7–10]: the theory possesses a number of *propagating torsion modes* (tordions) and black hole solution, its (A)dS sector is characterized by well-defined conserved charges and central charges, the existence of torsion is compatible with the AdS/CFT correspondence, and the canonical structure shows a close resemblance with the four-dimensional theory.

In the present paper, we continue studying dynamical aspects of the quadratic PGT in 3D by looking for *exact wave solutions with torsion*. The weak-field approximation of Einstein’s theory around the Minkowski background leads to a simple picture of the wave nature of gravity, which is recognized to have a striking analogy to the electromagnetic phenomena [11, 12]. By giving a covariant formulation of this analogy, one can generalize the linearized gravitational wave to the concept of an exact wave solution of general relativity [13–15]. Here, in the context of the quadratic PGT, such generalizations are used to find a class of exact wave solutions with torsion.

A gravitational wave with torsion in 3D was first found by Obukhov [16], in the framework of the Mielke–Baekler model [2]. Since the model is defined by a topological action, it was necessary to introduce matter, chosen in the form of Maxwell field, to have a nontrivial wave solution. On the other hand, our wave solution, being an exact vacuum solution of the

quadratic PGT, offers a new insight into the wave structure of genuine gravitational degrees of freedom, the propagating torsion modes.

The paper is organized as follows. In section 2, we give an overview of the plane-fronted gravitational waves in general relativity without/with gravitational constant, denoted shortly as GR/GR_Λ, as a basis for further extension to torsion waves in the quadratic PGT. In section 3, we start with the GR_Λ form of the metric and introduce a convenient ansatz for the RC connection, or equivalently, for the torsion. The only irreducible component of torsion is taken to be its tensorial piece, parametrized by a single function K . Then, we find the PGT field equations that impose dynamical restrictions on K . A characteristic parameter appearing in these equations is the mass parameter μ^2 , associated to the torsion spin-2 mode. In sections 4 and 5, we find a class of exact torsion waves and classify them according to the values of two parameters, μ^2 and λ , the latter one being related to the value of cosmological constant. In section 6, we discuss criteria that are used to recognize the wave nature of exact solutions and conclude with some specific remarks. Finally, two appendices contain a useful technical information.

Our conventions are the same as in Ref. [8]: the Latin indices (i, j, k, \dots) refer to the local Lorentz frame, the Greek indices (μ, ν, ρ, \dots) refer to the coordinate frame, and both run over 0,1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor ε^{ijk} is normalized to $\varepsilon^{012} = 1$; b^i is the orthonormal triad (coframe 1-form), h_i is the dual basis (frame), the Hodge dual of a form α is ${}^*\alpha$, and the exterior product of forms is implicit.

II. PLANE-FRONTED WAVES IN GENERAL RELATIVITY

In this section, we give a short account of the plane-fronted gravitational waves as exact solutions of Einstein's general relativity.

A. pp-waves in GR

A specific class of plane-fronted waves, characterized by having parallel rays (pp-waves for short), can be described, in suitable local coordinates, by the metric [13–15]

$$ds^2 = H(u, y)du^2 + 2dudv - dy^2, \quad (2.1)$$

where u is interpreted as the phase of the wave and ∂_v is the covariantly constant null vector field. This metric is a natural generalization of the *linearized* gravitational plane waves propagating on the background Minkowski spacetime [11, 12]. General criteria for identifying the wave nature of *exact* solutions will be discussed in section VI.

The explicit form of $H(u, y)$ in (2.1) can be determined by the GR field equations. Since the only nonvanishing component of the Ricci tensor is $(Ric)_{uu} = H''/2$ (prime means differentiation with respect to y) and the scalar curvature identically vanishes, $R = 0$, the *vacuum* field equations of GR imply

$$H'' = 0 \quad \Rightarrow \quad H = h_1(u) + h_2(u)y, \quad (2.2)$$

where h_1, h_2 are the integration “constants”. This solution is in fact trivial since for $H'' = 0$ the Ricci tensor vanishes and, in 3D, the full curvature tensor also vanishes. Hence, (2.2) defines a Minkowski spacetime in nonstandard coordinates.

Thus, in GR, nontrivial pp-waves can exist only in the presence of *matter*; see, for instance [17–19]. Note, however, that true *vacuum* waves can exist also in *new dynamical settings*, such as Topologically massive gravity or New massive gravity [19–21]. The vacuum waves are an idealization of wave solutions in the region far from matter sources.

B. Plane-fronted waves in GR_Λ

Now, we turn to a generalized dynamical framework of GR_Λ by allowing a nonvanishing cosmological constant. The pp-wave (2.1) is not a vacuum solution of GR_Λ . Indeed, the fact that $R = 0$ for the metric (2.1) implies $\Lambda = 0$. A plane-fronted wave that is compatible with $\Lambda \neq 0$ can be conveniently represented by the metric

$$ds^2 = 2 \left(\frac{q}{p} \right)^2 du (S du + dv) - \frac{dy^2}{p^2}, \quad (2.3a)$$

see Ozsváth [22] and Obukhov [23], where the functions p, q and S are chosen as [16]

$$p = 1 + \frac{\lambda}{4}y^2, \quad q = 1 - \frac{\lambda}{4}y^2, \quad S = -\frac{\lambda}{2}v^2 + \frac{\sqrt{p}}{2q}H(u, y). \quad (2.3b)$$

Clearly, the limit $\lambda = 0$ returns us back to the pp-wave (2.1). Introducing the orthonormal triad field as

$$\begin{aligned} b^0 &:= \frac{1}{\sqrt{2}} \left[\left(1 + \frac{q^2}{p^2} S \right) du + \frac{q^2}{p^2} dv \right], \\ b^1 &:= \frac{1}{\sqrt{2}} \left[\left(1 - \frac{q^2}{p^2} S \right) du - \frac{q^2}{p^2} dv \right], \\ b^2 &:= \frac{1}{p} dy, \end{aligned} \quad (2.4)$$

the metric can be written as $ds^2 = \eta_{ij} b^i \otimes b^j$, with $\eta_{ij} = \text{diag}(+1, -1, -1)$. In the literature, one often uses the light-cone components of the triad:

$$b^+ := du, \quad b^- := \frac{q^2}{p^2} (S du + dv).$$

In order to verify that the triad (2.4) satisfies the GR_Λ field equations,

$$a_0 \left((Ric)^i - \frac{1}{2} R b^i \right) - \Lambda b^i = 0, \quad a_0 := \frac{1}{16\pi G}, \quad (2.5)$$

we first calculate the Christoffel connection; it has the form

$$\begin{aligned} \Gamma^{01} &= \frac{\lambda y}{q} b^2 - \frac{\lambda v}{\sqrt{2}} (b^0 + b^1), \\ \Gamma^{02} &= \frac{\lambda y}{q} b^0 - \frac{1}{2} (b^0 + b^1) (q^2 S' / p), \\ \Gamma^{12} &= \frac{\lambda y}{q} b^1 + \frac{1}{2} (b^0 + b^1) (q^2 S' / p), \end{aligned}$$

or, more compactly:

$$\Gamma^{ij} = \bar{\Gamma}^{ij} + \frac{1}{2} \varepsilon^{ij}{}_m k^m k_n b^n (q^2 S' / p). \quad (2.6)$$

Here, the first term, $\bar{\Gamma}^{ij} := \Gamma^{ij}(S' = 0)$, is the piece that describes the “background” (A)dS geometry of spacetime, whereas the second term is the radiation piece, characterized by the null vector $k^i = (1, -1, 0)$, $k^2 = 0$, which is not covariantly constant for $\lambda \neq 0$.

Next, we calculate the curvature $R^{ij} = d\Gamma^{ij} + \Gamma^i{}_m \Gamma^{mj}$,

$$R^{ij} = -\lambda b^i b^j + \varepsilon^{ijm} k_m k_n b^n p (q^2 S' / p)', \quad (2.7a)$$

where ${}^*b_n = (1/2) \varepsilon_{nrs} b^r b^s$. Note that the radiation piece of R^{ij} is clearly separated from the (A)dS piece. Finally, the form of the Ricci curvature $(Ric)^i = -h_j \lrcorner R^{ij}$ and the scalar curvature $R = h_i \lrcorner (Ric)^i$,

$$\begin{aligned} (Ric)^i &= -2\lambda b^i + \frac{1}{2} k^i k_m b^m p (q^2 S' / p)', \\ R &= -6\lambda, \end{aligned} \quad (2.7b)$$

implies that the content of the field equations (2.5) is given by

$$\begin{aligned} a_0 \lambda &= A, & p \left(\frac{q^2}{p} S' \right)' &= 0, \\ \Rightarrow \quad \frac{\sqrt{p}}{2q} H &= \beta_1(u) + \beta_2(u) \frac{y}{q}. \end{aligned} \tag{2.8}$$

The function H defines the vacuum solution for the metric (2.3). Since the on-shell value of the curvature is $R^{ij} = -\lambda b^i b^j$, the geometry of the solution (2.8) is fixed: for $\lambda = 0$, > 0 , or < 0 , it has the Minkowskian, AdS, or dS form, respectively.

Thus, again, in order for the plane-fronted wave (2.3) to be a nontrivial exact solution, one has to introduce matter. However, by going over to PGT, we expect the new gravitational dynamics to allow for the existence of true wave solutions even in vacuum.

III. DYNAMICS OF TORSION WAVES

In this section, we briefly recapitulate basic aspects of PGT, introduce a geometric extension of the Riemannian plane-fronted waves (2.3) to torsion waves, and discuss their dynamics.

A. Basic aspects of PGT

The PGT is a gauge theory of gravity based on gauging the Poincaré group, with an underlying Riemann–Cartan (RC) geometry of spacetime [4–6]. Basic gravitational variables are the triad field b^i and the Lorentz connection $A^{ij} = -A^{ji}$ (1-forms), and the corresponding field strengths are the torsion $T^i = db^i + A^i_k b^k$ and the curvature $R^{ij} = dA^{ij} + A^i_k A^{kj}$ (2-forms). General dynamics of PGT is defined by the gravitational Lagrangian $L_G = L_G(b^i, T^i, R^{ij})$ (3-form). Varying L_G with respect to b^i and A^{ij} yields the respective gravitational field equations in vacuum [8],

$$\begin{aligned} (1\text{ST}) \quad \nabla H_i + E_i &= 0, \\ (2\text{ND}) \quad \nabla H_{ij} + E_{ij} &= 0, \end{aligned} \tag{3.1}$$

where

$$H_i := \frac{\partial L_G}{\partial T^i}, \quad H_{ij} := \frac{\partial L_G}{\partial R^{ij}},$$

are the covariant field momenta, and

$$E_i := \frac{\partial L_G}{\partial b^i}, \quad E_{ij} := \frac{\partial L_G}{\partial A^{ij}},$$

are the gravitational energy-momentum and spin currents. We require L_G to be parity invariant and at most quadratic in the field strengths. In that case, H_i and H_{ij} can be expressed linearly in terms of the irreducible pieces of the field strengths (Appendix A),

$$\begin{aligned} H_i &= 2^* (a_1^{(1)} T_i + a_2^{(2)} T_i + a_3^{(3)} T_i), \\ H_{ij} &= -2a_0 \varepsilon_{ijk} b^k + H'_{ij}, \\ H'_{ij} &:= 2^* (b_4^{(4)} R_{ij} + b_5^{(5)} R_{ij} + b_6^{(6)} R_{ij}), \end{aligned} \tag{3.2a}$$

where a_0, a_n and b_n are coupling constants; moreover, the gravitational Lagrangian takes the form

$$L_G = \frac{1}{2} T^i H_i + R^{ij} (-a_0 \varepsilon_{ijk} b^k) + \frac{1}{4} R^{ij} H'_{ij} - \frac{1}{3} A_0 \varepsilon_{ijk} b^i b^j b^k, \tag{3.2b}$$

and the gravitational energy-momentum and spin currents turn out to be

$$\begin{aligned} E_i &= h_i \lrcorner L_G - (h_i \lrcorner T^m) H_m + \frac{1}{4} (h_i \lrcorner R^{mn}) H_{mn}, \\ E_{ij} &= -(b_i H_j - b_j H_i). \end{aligned} \tag{3.2c}$$

B. Geometry of the ansatz

In our search for the generalized plane-fronted waves, we assume that the form of the triad field (2.4) remains unchanged, whereas the connection is determined by the following rule:

- Starting with the Riemannian connection (2.6), (i) we leave its first, (A)dS piece $\bar{\Gamma}^{ij}$ unchanged, (ii) but modify the second, radiation piece in a way that *preserves* the wave nature of the solution.

The instruction (ii) is realized by adopting the following ansatz for the RC connection:

$$A^{ij} = \bar{\Gamma}^{ij} + \frac{1}{2} \varepsilon^{ij}{}_m k^m k_n b^n G, \tag{3.3a}$$

$$G := \frac{q^2}{p} (S' + K). \tag{3.3b}$$

Here, the new term $K = K(u, y)$ describes the effect of torsion, as follows from

$$T^i := \nabla b^i = \frac{q^2}{2p} K k^i k_m {}^\star b^m. \quad (3.4)$$

The only nonvanishing irreducible piece of T^i is its tensorial piece (Appendix A):

$${}^{(1)}T^i = T^i.$$

Having chosen the form of the connection, one can now calculate the RC curvatures; they are obtained from (2.7) by the replacement $S' \rightarrow S' + K$:

$$\begin{aligned} R^{ij} &= -\lambda b^i b^j + \varepsilon^{ijm} k_m k^n {}^\star b_n p G', \\ (Ric)^i &= -2\lambda b^i + \frac{1}{2} k^i k_m b^m p G', \\ R &= -6\lambda. \end{aligned} \quad (3.5)$$

The nonvanishing irreducible components of the curvature R^{ij} are (Appendix A)

$${}^{(4)}R^{ij} = \frac{1}{2} \varepsilon^{ijm} k_m k^n {}^\star b_n p G', \quad {}^{(6)}R^{ij} = -\lambda b^i b^j,$$

and the quadratic curvature invariant has the form $R^{ij} {}^\star R_{ij} = 6\lambda^2 {}^\star 1$.

The geometric configuration defined by the triad field (2.4) and the connection (3.3) represents a generalized gravitational plane-fronted wave of GR_Λ , or the *torsion wave* for short. More details on its wave nature will be given in section VI.

C. Field equations

Having found the expressions for the torsion and the curvature, one can now calculate the covariant momenta H_i, H_{ij} , and the energy-momentum and spin currents E_i, E_{ij} , and obtain the explicit form of the PGT field equations (3.1). The result takes the following form [24]:

$$\begin{aligned} (1ST) \quad & (a_0 + b_4 \lambda + b_6 \lambda) p G' - a_1 q (q K)' = 0, \\ & 2\Lambda - 2a_0 \lambda + b_6 \lambda^2 = 0, \\ (2ND) \quad & b_4 (2G'' p^3 q + G' \lambda y p^3 + 2G' \lambda y p^2 q) + 2(a_1 - a_0 - b_6 \lambda) K q^3 = 0. \end{aligned} \quad (3.6)$$

The second equation in (1ST) defines a relation between the parameter λ of the solution and the coupling constants. For $b_6 = 0$, it takes a particularly simple form: $a_0 \lambda = \Lambda$. By

noting that $(2ND)$ can be rewritten as

$$2b_4p[pq(pG')' + (pG')\lambda y] + 2(a_1 - a_0 - b_6\lambda)Kq^3 = 0,$$

one finds that the field equations (3.6) can be transformed to a more compact form:

$$\begin{aligned} (1ST) \quad pG' &= C_0q\mathcal{K}', & C_0 &= \frac{a_1}{a_0 + (b_4 + b_6)\lambda}, \\ (2ND) \quad p(p\mathcal{K}')' + \mu^2\mathcal{K} &= 0, & \mu^2 &= \frac{a_1 - a_0 - b_6\lambda}{b_4C_0}, \end{aligned} \quad (3.7)$$

with $\mathcal{K} := qK$.

In PGT, the spectrum of excitations around the Minkowski spacetime consists of 6 independent torsion modes: one scalar, one pseudoscalar, two spin-1 and two spin-2 states [7, 8]. Two spin-2 states form a parity invariant multiplet associated to the tensorial piece of the torsion, with equal masses: $m^2 = a_0(a_1 - a_0)/(a_1b_4)$. Since our ansatz (3.4) reduces torsion just to its tensorial piece, it is not surprising that for $\lambda = 0$, the coefficient μ^2 in (3.7) reduces exactly to m^2 . For $\lambda \neq 0$, μ^2 is associated to the spin-2 excitations around the (A)dS background, and the condition for the absence of tachions requires $\mu^2 \geq 0$.

In what follows, we will solve two dynamical equations (3.7) for the unknown functions \mathcal{K} and G , assuming $\mu^2 \geq 0$; then, we will use (3.3b) to find S . The torsion function K and the metric function S , obtained in this way, completely define the solution.

IV. MASSIVE TORSION WAVES

In this section, we classify the solutions of the field equations (3.7) for $\mu^2 > 0$, according to the values of λ .

A. $\lambda = 0$

The simplest form of equations (3.7) is obtained in the limit $\lambda \rightarrow 0$:

$$\begin{aligned} a_0G' - a_1K' &= 0, & \Lambda &= 0, \\ K'' + m^2K &= 0, & m^2 &= \frac{a_0(a_1 - a_0)}{b_4a_1}, \end{aligned} \quad (4.1)$$

with $G = S' + K$ and $S = H/2$. The solution has a simple form:

$$\begin{aligned} K &= A(u) \cos my + B(u) \sin my, \\ \frac{1}{2}H &= \frac{a_1 - a_0}{a_0 m} (A \sin my - B \cos my) + h_1(u) + h_2(u)y. \end{aligned} \quad (4.2)$$

In Riemannian gravity, one can remove the term $h_1 + h_2 y$ in H by a coordinate transformation. This transformation does not change the form of the metric (2.1), which is the only dynamical variable of the theory in vacuum. In the RC theory, such a coordinate transformation is not particularly useful as it affects the form of the connection. Note, however, that the term $h_1 + h_2 y$ has no influence upon the RC curvature, which depends only on H'' . Thus, without loss of generality, we can choose $h_1 = h_2 = 0$.

The vector field $k = \partial_v$ is the Killing vector for both the metric and the torsion; moreover, it is a null and covariantly constant vector field. This allows us to consider the solution (4.2) as a generalized pp-wave.

B. $\lambda > 0$

For positive λ , we use the notation

$$\lambda = \frac{1}{\ell^2}, \quad x = \frac{y}{2\ell}, \quad \kappa = 2\mu\ell,$$

so that $\int dy = 2\ell \int dx$. Now, having in mind the form of the solution (4.2) for $\lambda = 0$, we use a similar ansatz for the torsion function $\mathcal{K} \equiv qK$:

$$\mathcal{K} = A \cos \alpha + B \sin \alpha, \quad \alpha = \alpha(y), \quad (4.3a)$$

where $A = A(u)$, $B = B(u)$. Substituting this into (2ND) of (3.7) produces two conditions on α :

$$p^2(\alpha')^2 - \mu^2 = 0, \quad p^2\alpha'' - \frac{1}{2}\lambda y p\alpha' = 0.$$

The first condition yields

$$\alpha' = \frac{\mu}{p} = \frac{\mu}{1+x^2} \quad \Rightarrow \quad \alpha = 2\ell \int \frac{\mu}{1+x^2} dx = \kappa \arctan x, \quad (4.3b)$$

whereas the second one is automatically satisfied. In the limit $\lambda \rightarrow 0$, we have $\alpha \rightarrow \kappa x = my$, and (4.3) reduces to (4.2).

In the next step, we use (4.3) and (1ST) to calculate G :

$$G = 2\ell C_0 \int \frac{q}{p} \mathcal{K}' dx = D \frac{1}{p} \left[\left(qA - \frac{4x}{\kappa} B \right) \cos \alpha + \left(qB + \frac{4x}{\kappa} A \right) \sin \alpha \right],$$

where $D = C_0 \kappa^2 / (\kappa^2 - 4)$. Finally, integrating the relation $S' = (p/q^2)G - K$ yields the metric function H . Using the definition

$$\mathcal{H} := \frac{\sqrt{p}}{2q} H \equiv S + \frac{\lambda}{2} v^2, \quad (4.4)$$

we find:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_1 + \mathcal{H}_2, \\ \mathcal{H}_1 &:= 2\ell \int \frac{p}{q^2} G dx = 2\ell D \cdot \frac{p}{\kappa q} (A \sin \alpha - B \cos \alpha), \\ \mathcal{H}_2 &:= -2\ell \int K dx = \frac{2\ell}{\kappa^2 - 4} \times \\ &\quad \left[(B - iA)(2 + \kappa) e^{i(2-\kappa) \arctan x} {}_2F_1 \left(1, \frac{2-\kappa}{4}; \frac{6-\kappa}{4}; -e^{4i \arctan x} \right) \right. \\ &\quad \left. - (B + iA)(2 - \kappa) e^{i(2+\kappa) \arctan x} {}_2F_1 \left(1, \frac{2+\kappa}{4}; \frac{6+\kappa}{4}; -e^{4i \arctan x} \right) \right], \end{aligned} \quad (4.5)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function [24]. Here, again, the integration term $h_1(u) + h_2(u)y/q$ appearing in \mathcal{H} is removed, as it has no influence upon the RC curvature.

In order to illustrate the form of the torsion wave, we display here the plots of the torsion function $(q^2/p)K(u, y)$ and the curvature function $pG'(u, y)/2$, for a specific choice of the parameters ℓ, κ , and for fixed amplitudes $A(u)$ and $B(u)$.

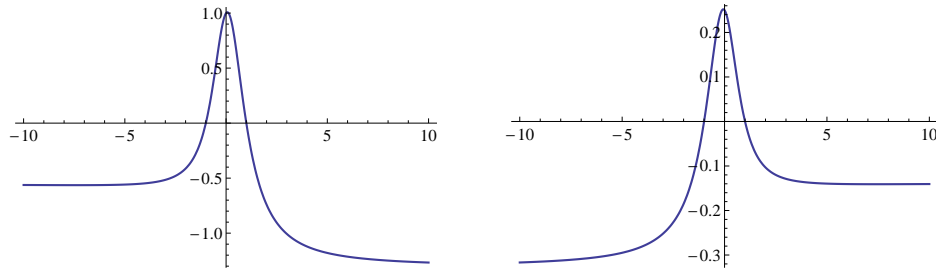


FIG. 1: The form of the torsion function $(q^2/p)K$ (left) and the curvature function $pG'/2$ (right) for $\mu^2 > 0$, in the region $x \in [-10, 10]$, and for $A(u) = B(u) = 1$, $\ell = 1$, $\kappa = 1/4$.

C. $\lambda < 0$

In this case, we use the notation

$$\lambda = -\frac{1}{\ell^2}, \quad x = \frac{y}{2\ell}, \quad \kappa = 2\ell\mu,$$

and find that the torsion function \mathcal{K} is given by

$$\mathcal{K} = A \cos \alpha + B \sin \alpha, \quad \alpha = \kappa \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| = \kappa \operatorname{arctanh} x. \quad (4.6)$$

Here, $\alpha(x)$ is singular at $x = 1$, but for $\lambda \rightarrow 0$, it has the expected limit: $\alpha \rightarrow \kappa x = my$.

Then, following the same steps as in the previous subsection, we can first calculate G ,

$$G = \frac{E}{p} \left[\left(Bq - \frac{4x}{\kappa} A \right) \sin \alpha + \left(Aq + \frac{4x}{\kappa} B \right) \cos \alpha \right],$$

where $E = C_0 \kappa^2 / (\kappa^2 + 4)$, and then find the metric function \mathcal{H} :

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_1 + \mathcal{H}_2, \\ \mathcal{H}_1 &:= 2\ell \int \frac{p}{q^2} G dx = 2\ell \frac{E}{\kappa} \frac{p}{q} [A \sin \alpha - B \cos \alpha], \\ \mathcal{H}_2 &:= -2\ell \int K dx = -\frac{2\ell i}{\kappa^2 + 4} \times \\ &\quad \left[(B - iA)(2 + i\kappa) e^{(2-i\kappa) \operatorname{arctanh} x} {}_2F_1 \left(1, \frac{2-i\kappa}{4}; \frac{6-i\kappa}{4}; -e^{4 \operatorname{arctanh} x} \right) \right. \\ &\quad \left. - (B + iA)(2 - i\kappa) e^{(2+i\kappa) \operatorname{arctanh} x} {}_2F_1 \left(1, \frac{2+i\kappa}{4}; \frac{6+i\kappa}{4}; -e^{4 \operatorname{arctanh} x} \right) \right]. \quad (4.7) \end{aligned}$$

As before, all the integration terms in \mathcal{H} are removed.

This solution can be obtained from the one for $\lambda > 0$ by the analytic continuation in ℓ :

$$\ell \rightarrow i\ell \quad \Rightarrow \quad \kappa \rightarrow i\kappa, \quad x \rightarrow \frac{1}{i}x, \quad \arctan x \rightarrow \frac{1}{i} \operatorname{arctanh} x.$$

For the asymptotic behavior of both massive and massless torsion waves, see section VI and Appendix B.

V. MASSLESS TORSION WAVES

For $\mu^2 = 0$, we have $a_1 - a_0 - b_6 \lambda = 0$ and the field equations (3.7) are simplified:

$$pG' = C_0 q \mathcal{K}', \quad p(p\mathcal{K}')' = 0. \quad (5.1)$$

A. $\lambda = 0$

For vanishing λ , the field equations with $C_0 = 1$ take the form:

$$G' - K' \equiv \frac{1}{2}H'' = 0, \quad K'' = 0, \quad (5.2)$$

so that

$$H = h_1(u) + h_2(u)y, \quad K = k_1(u) + k_2(u)y. \quad (5.3)$$

This is a rather strange solution: since the metric function H is trivial, the metric takes the Minkowski form and consequently, it is dynamically decoupled from the torsion.

B. $\lambda > 0$

For the positive cosmological constant, with $\lambda := 1/\ell^2$ and $x = y/2\ell$, the solution reads:

$$\begin{aligned} \mathcal{K} &= A(u) \arctan x + B(u), \\ G &= A(u) \frac{C_0 x}{p}, \\ \mathcal{H}(u, y) &= \ell A(u) \left(\frac{C_0}{q} - \arctan x \cdot \ln \frac{1 - ie^{2i \arctan x}}{1 + ie^{2i \arctan x}} \right) \\ &\quad + \frac{i\ell}{2} A(u) \left[\text{Li}_2\left(ie^{2i \arctan x}\right) - \text{Li}_2\left(-ie^{2i \arctan x}\right) \right] \\ &\quad - 2\ell B(u) \operatorname{arctanh} x, \end{aligned} \quad (5.4)$$

where $\text{Li}_2(z)$ is the dilogarithm function [24]. The solution is illustrated in Fig. 2.

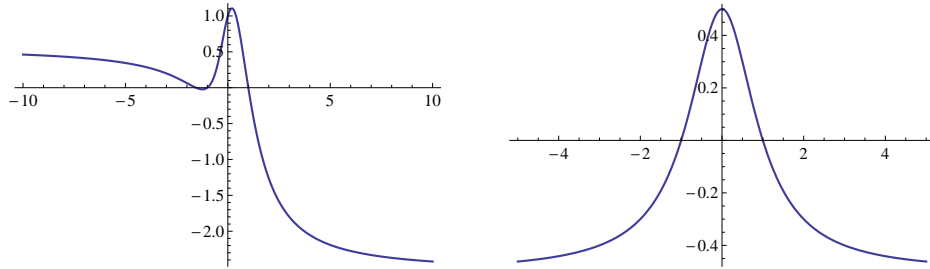


FIG. 2: The form of the torsion function $(q^2/p)K$ (left) and the curvature function $pG'/2$ (right) for $\mu^2 = 0$, in the region $x \in [-10, 10]$, and for $A(u) = B(u) = 1$ and $\ell = 1$.

C. $\lambda < 0$

Finally, for $\lambda := -1/\ell^2$, one finds:

$$\begin{aligned}
\mathcal{K} &= A(u) \operatorname{arctanh} x + B(u), \\
G &= A(u) \frac{C_0 x}{p}, \\
\mathcal{H}(u, y) &= \ell A(u) \left(-\frac{C_0}{q} - \frac{i}{2} \operatorname{arctanh} x \cdot \ln \frac{1 - ie^{-2 \operatorname{arctanh} x}}{1 + ie^{-2 \operatorname{arctanh} x}} \right) \\
&\quad + \frac{i\ell}{2} A(u) \left[\operatorname{Li}_2 \left(ie^{-2 \operatorname{arctanh} x} \right) - \operatorname{Li}_2 \left(-ie^{-2 \operatorname{arctanh} x} \right) \right] \\
&\quad + 2\ell B(u) \operatorname{arctan} x.
\end{aligned} \tag{5.5}$$

VI. DISCUSSION AND CONCLUSIONS

In this paper, we derived a new class of exact solutions of 3D gravity with propagating torsion in empty spacetime, the generalized plane-fronted waves, or the torsion waves.

The wave ansatz for the metric (2.4) and the RC connection (3.3) represents a natural generalization of the Riemannian plane-fronted waves with cosmological constant. However, a covariant characterization of the wave nature of an exact solution is a rather complex issue [13–15], which has not been fully clarified for non-Riemannian theories of gravity; for an attempt in this direction, see [25].

The existence of the null covector $k_i = (1, 1, 0)$, appearing already in the RC connection (3.3), is an essential element of the geometric structure of a gravitational wave. It can be represented as the 1-form $k_i b^i = \sqrt{2} du$, associated to the wave fronts $u = \text{const}$. The related vector field $k^i \partial_i = \sqrt{2} \partial_u$ is orthogonal to the y direction; moreover, for $\lambda = 0$, k^i is covariantly constant (pp-wave).

Based on an analogy with the electromagnetism, Lichnerowicz proposed a covariant criterion for the existence of gravitational waves in general relativity, see [14]. After separating the radiation piece of the RC curvature (3.5), $S^{ij} := R^{ij} + 2\lambda b^i b^j$, one can verify that it satisfies Lichnerowicz's requirements:

$$k^i S_{ijmn} = 0, \quad \varepsilon^{ijk} k_i S_{jkmn} = 0. \tag{6.1}$$

Clearly, the above criterion is not sufficient for a RC geometry, where we have one more field strength, the torsion. However, in analogy with electromagnetism, the radiation conditions

for torsion are expected to have the form

$$k^m T_{imk} = 0, \quad \varepsilon^{mnk} k_m T_{ink} = 0. \quad (6.2)$$

A direct verification based on (3.4) shows that these conditions are also satisfied. The radiation properties (6.1) and (6.2) strongly support the interpretation of our generalized plane-fronted wave as a genuine PGT extension of the related Riemannian structure.

One should also note that our RC curvature has the same irreducible components as the corresponding Riemannian curvature, and moreover, it has all the usual index symmetries of the Riemannian curvature; in particular, $R_{ijmn} = R_{mni j}$. The same properties were found by Pašić and Vassiliev [26] in their pp-wave with torsion, constructed in the model with metric-compatible connection and curvature squared Lagrangian. The torsion of their solution is pure tensor, as in our case.

In electrodynamics and in general relativity, exact wave solutions are associated to massless modes of the related fields, so that the appearance of massive torsion waves may seem a bit strange. However, the existence of massive torsion modes is not in conflict with the gauge structure of PGT, it is a generic feature associated to the presence of T^2 terms in the Lagrangian. Massive waves appear also in some Riemannian extensions of GR, such as Topologically massive gravity or New massive gravity [19–21].

Asymptotic properties of the torsion waves are defined by the large y limits of the torsion (3.4) and the RC curvature (3.5). As follows from the results of Appendix B, the generic asymptotic form of the torsion waves does not coincide with the (A)dS geometry.

Our study of exact torsion waves in 3D can be considered as a complement to the related results in four dimensions [25–27]. In particular, we wish to place emphasis on the results of Sippel and Goenner [25], who made a significant progress in clarifying the structure of pp-waves with torsion: (i) they generalized the Ehlers–Kundt classification of pp-waves [13] by relaxing the assumption that the GR field equations hold, and (ii) they introduced a classification of the allowed form of torsion in pp-waves. Further advances in this direction would help us to better understand the role of torsion in exact wave solutions.

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Appendix A: Irreducible decomposition

For the sake of completeness, we present here the form of the irreducible components of T^i and R^{ij} , see also [8], with the wedge product sign explicitly displayed.

Torsion has three irreducible components, the vector, axial and tensor component:

$$\begin{aligned} {}^{(2)}T_i &:= \frac{1}{2}b_i \wedge (h_m \lrcorner T^m) = \frac{1}{2}\eta_{ij}V_k b^j \wedge b^k, \\ {}^{(3)}T_i &:= \frac{1}{3}\star[b_i \wedge \star(T^m \wedge b_m)] = \frac{1}{2}A\varepsilon_{ijk}b^j \wedge b^k, \\ {}^{(1)}T_i &:= T_i - {}^{(2)}T_i - {}^{(3)}T_i, \end{aligned} \tag{A1}$$

where $V_k := T^m{}_{mk}$ and $A := \varepsilon_{ijk}T^{ijk}/6$.

The curvature also has three irreducible pieces. Making use of the definitions

$$\begin{aligned} A_i &:= \frac{1}{2}h_i \lrcorner (b^k \wedge \hat{R}_k) = \hat{R}_{[ik]}b^k, \\ \mathcal{S}_i &:= \hat{R}_i - A_i - \frac{1}{3}Rb_i = \hat{R}_{(ik)}b^k - \frac{1}{3}Rb_i, \end{aligned}$$

where $\hat{R}_i := (Ric)_i$, the irreducible pieces of R_{ij} read:

$$\begin{aligned} {}^{(4)}R_{ij} &:= b_i \mathcal{S}_j - b_j \mathcal{S}_i, \\ {}^{(5)}R_{ij} &:= b_i A_j - b_j A_i, \\ {}^{(6)}R_{ij} &:= \frac{1}{6}Rb_i \wedge b_j. \end{aligned} \tag{A2}$$

Note that in 3D, the Weyl curvature vanishes.

Appendix B: Asymptotic geometry

In this appendix, we calculate the large y limits of the expressions $(q^2/p)K$ and $pG'/2$; these limits define the respective asymptotic values of the torsion and the radiation piece of the curvature, characterizing the gravitational wave. The formulas for $\lambda = 0$ are omitted, as the related asymptotic behavior can be read off directly from the main text.

Case $\mu^2 > 0$

$\lambda > 0$:

$$\begin{aligned}\lim_{y \rightarrow \pm\infty} \frac{q^2}{p} K &= - \left(A \cos \frac{\kappa\pi}{2} \pm B \sin \frac{\kappa\pi}{2} \right), \\ \lim_{y \rightarrow \pm\infty} \frac{1}{2} p G' &= \frac{1}{2} C_0 \mu \left(\pm A \sin \frac{\kappa\pi}{2} - B \cos \frac{\kappa\pi}{2} \right).\end{aligned}\tag{B1}$$

$\lambda < 0$:

$$\begin{aligned}\lim_{y \rightarrow \infty} \frac{q^2}{p} K &= -A, \\ \lim_{y \rightarrow \infty} \frac{1}{2} p G' &= -\frac{1}{2} C_0 \mu B.\end{aligned}\tag{B2}$$

Case $\mu^2 = 0$

$\lambda > 0$:

$$\begin{aligned}\lim_{y \rightarrow \pm\infty} \frac{q^2}{p} K &= \mp A \frac{\pi}{2} - B, \\ \lim_{y \rightarrow \infty} \frac{1}{2} p G' &= -\frac{AC_0}{4\ell}.\end{aligned}\tag{B3}$$

$\lambda < 0$:

$$\begin{aligned}\lim_{y \rightarrow \infty} \frac{q^2}{p} K &= -B, \\ \lim_{y \rightarrow \infty} \frac{1}{2} p G' &= -\frac{AC_0}{4\ell}.\end{aligned}\tag{B4}$$

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- [1] For a review and an extensive list of references, see: S. Carlip, *Quantum Gravity in 2+1 Dimensions* (Cambridge University Press, Cambridge, 1998); Conformal field theory, (2+1)-dimensional gravity, and the BTZ black hole, *Class. Quantum Grav.* **22**, R85–R124 (2005).
- [2] E. W. Mielke and P. Baekler, Topological gauge model of gravity with torsion, *Phys. Lett. A* **156**, 399–403 (1991).
- [3] T. W. B. Kibble, Lorentz invariance and the gravitational field, *J. Math. Phys.* **2**, 212–221 (1961); D. W. Sciama, On the analogy between charge and spin in general relativity, in: *Recent Developments in General Relativity, Festschrift for Infeld* (Pergamon Press, Oxford; PWN, Warsaw, 1962) pp. 415–439.

- [4] For a textbook exposition of PGT, see: M. Blagojević, *Gravitation and Gauge Symmetries* (Institute of Physics, Bristol, 2002); T. Ortín, *Gravity and Strings* (Cambridge University Press, Cambridge, 2004).
- [5] An up-to-date status of PGT, including its 3D version, can be found in: M. Blagojević and F. W. Hehl (eds.), *Gauge Theories of Gravitation, A Reader with Commentaries* (Imperial College Press, London, 2013).
- [6] Yu. N. Obukhov, Poincaré gauge gravity: Selected topics, *Int. J. Geom. Meth. Mod. Phys.* **3**, 95–138 (2006).
- [7] J. A. Helayël-Neto, C. A. Hernaski, B. Pereira-Dias, A. A. Vargas-Paredes, and V. J. Vasquez-Otoya, Chern-Simons gravity with (curvature)²- and (torsion)²-terms and a basis of degree-of-freedom projection operators, *Phys. Rev. D* **82**, 064014 (2010).
- [8] M. Blagojević and B. Cvetković, 3D gravity with propagating torsion: the AdS sector, *Phys. Rev. D* **85**, 104003 (2012) [10 pages].
- [9] M. Blagojević, B. Cvetković, O. Mišković and R. Olea, Holography in 3D AdS gravity with torsion, *JHEP* **1305** (2013) 103 [29 pages];
- [10] M. Blagojević, B. Cvetković, 3D gravity with propagating torsion: Hamiltonian structure of the scalar sector, *Phys. Rev. D* **85**, 104032 (2013) [15 pages].
- [11] H. Stephani, *Relativity, An Introduction to Special and General Relativity* (Cambridge University Press, Cambridge, UK, 2004), chapter 29.
- [12] A. Peres, Some gravitational waves, *Phys. Rev. Lett.* **3**, 571–572 (1959); reprinted in arXiv:hep-th/0205040.
- [13] J. Ehlers and W. Kundt, Exact solutions of the gravitational field equations, in: *Gravitation: an introduction to current research* (Wiley, N.Y., 1962), ed. L. Witten, pp. 49–101.
- [14] V. Zakharov, *Gravitational waves in Einstein's theory* (Halsted Press, New York, 1973).
- [15] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einsteins field equations*, 2nd ed. (Cambridge Univ. Press, Cambridge, 2003).
- [16] Y. Obukhov, New solutions in 3D gravity, *Phys. Rev. D* **68** 124015 (2003).
- [17] S. Deser, R. Jackiw and S.-Y. Pi, Cotton blend gravity pp waves, *Acta Phys. Polon. B* **36**, 27–34 (2005).
- [18] E. Ayón-Beato and M. Hassaïne, pp-waves of conformal gravity with self-interacting source, *Annals Phys.* **317**, 175–181 (2005);

- [19] E. Ayón-Beato and M. Hassaine, Scalar fields nonminimally coupled to pp-waves, *Phys. Rev. D* **71**, 084004 (2005).
- [20] E. Ayón-Beato, G. Giribet, and M. Hassaine, Bending AdS waves with New Massive Gravity, *JHEP* **0905** (2009) 029.
- [21] T. Moon and Y. S. Myung, Polarization modes of gravitational waves in three-dimensional massive gravities, *Phys. Rev. D* **85**, 027501 (2012).
- [22] I. Osváth, I. Robinson and K. Rózga, Plane-fronted gravitational and electromagnetic waves in spaces with cosmological constant, *J. Math. Phys.* **26**, 1755–1761 (1985).
- [23] Y. Obukhov, Generalized plane-fronted gravitational waves in any dimension, *Phys. Rev. D* **69**, 024013 (2004).
- [24] The field equations (3.6) for the torsion waves are checked using the *Excalc package* of the computer algebra system *Reduce*; after being transformed to the form (3.7), they are solved with the help of *Wolfram Mathematica*; see also:
Pocketbook of Mathematical Functions, abridged edition of *Handbook of Mathematical Functions*, M. Abramowitz and I. Stegun (eds.), material selected by M. Danos and F. Rafelski (Verlag Harri Deutsch, Frankfurt am Mein, FRG, 1984); chapters 15 (hypergeometric functions) and 27.7 (dilogarithm).
- [25] R. Sippel and H. Goenner, Symmetry classes of pp-waves, *Gen. Rel. Grav.* **18**, 1229–1243 (1986).
- [26] V. Pašić and D. Vassiliev, PP-waves with torsion and metric-affine gravity, *Class. Quant. Grav.* **22** 3961–3976 (2005).
- [27] W. Adamowicz, Plane waves in gauge theories of gravitation, *Gen. Rel. Grav.* **12**, 677–691 (1980); P. Singh and J. B. Griffiths, A new class of exact solutions of the vacuum quadratic Poincaré gauge field theory, *Gen. Rel. Grav.* **22**, 947–956 (1990); V. V. Zhytnikov, Wavelike exact solutions of $R + R^2 + Q^2$ gravity, *J. Math. Phys.* **35**, 6001–6017 (1994); M.-K. Chen, D.-C. Chern, R.-R. Hsu, and W. B. Yeung, Plane-fronted torsion waves in a gravitational gauge theory with a quadratic Lagrangian, *Phys. Rev. D* **28**, 2094–2095 (1983); O. V. Babourova, B. N. Frolov and E. A. Klimova, Plane torsion waves in quadratic gravitational theories, *Class. Quant. Grav.* **16**, 1149–1162 (1999). The wave solutions found in the last two references are rather peculiar: in both cases, the solutions for the metric and the torsion are dynamically decoupled.